STRUCTURAL ANALYSIS OF A GIRDER-ARCH BRIDGE

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SYNOPSIS

This report presents two methods, consistent deformations and an energy method, for the stress analysis of a girder-arch bridge. Using the first method, two solutions were calculated, the first of which takes into account the axial stress effects in the structure, and the second ignores these effects. A third solution is developed using the energy method.

For the method of consistent deformations, the first step is to find the deflections of the girder and of the arch separately. The redundant forces in the vertical member are then found by requiring the deformations of the girder, the arch and the vertical members to be consistent. The moment influence lines can then be drawn. The second method, which uses a minimum energy approach, treats the structure as a single system by assuming that the portion between the girder and the arch acts as one member. Using the virtual displacement method, the redundant reactions of the structure can then be found. Next, the total moment at each section of the compound structure can be calculated. Then the bending moment in the girder and in the arch can be found for each section by distributing the total moment induced in the compound section between the two elements. Using these results the moment influence lines for the girder and the arch can be drawn.

The computations in the report are generally expressed in matrix form and have been wholly performed by a computer so that the tediousness and possible inaccuracy of hand computation are minimized. A comparison of the three solutions indicates that the results are reasonably close to each other.

INTRODUCTION

The girder-arch bridge (Fig.1) analysed in this report is a mordern type of bridge which combines the girder and the arch by using vertical hinged members to connect the two main members. By combining the girder and the arch, the stiffness of the system is greater than that of a simple girder or a simple arch. The moment induced in the system is greatly reduced by the action of the arch. Furthermore, girder-arch structures are economical (1)[1] and are pleasing in appearance. This type of bridge is especially suitable for use as a long span over a deep valley.

Of the many possible methods of analysis for this type of bridge, this report presents two which may be used to analyse high-order statically indeterminate structures. A digital computer was available and therefore the calculations for both of these methods were wholly performed by the digital computer.

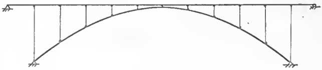


Fig. 1. Elevation of a girder-arch bridge.

[1]Numerals in parentheses refer to corresponding items in the References.

Since the bridge analysed in this report has variable cross sections in the arch, this report shows a general procedure for a computer analysis to handle a structure with variable cross sections. Because computers are now readily available, either of the methods presented in this report can be used to handle most structures with variable sections such as continuous beams or rigid frames very easily. Thus the methods this report presents are not restricted to this particular problem but can be generalized to analyse many indeterminate structures.

GENERAL ANALYSIS PROCEDURES

I. Consistent Deformation Method.

a. Solution 1.

Considering both flexural and axial stress effects.

This method requires that the deflections of the girder, the vertical truss members, and the arch be consistent at each node point when the structure is loaded. Therefore the first step is to find the deflections of the girder for each node. For drawing the influence lines, the load acting along the structure is taken as a moving unit load. This step is accomplished by using the conjugate beam method. The next step is to compute the deflections of the two-hinged arch. For this purpose, the deflections of the simple arch are first calculated, then the superposition method is applied to find the actual horizontal reactions of a two-hinged arch. Knowing the horizontal reactions and using the method of superposition again, the vertical deflections of a two-hinged arch are obtained. The third step is to set up the equations for calculating the redundant forces in the vertical members. Knowing the deflections of the girder and of the arch due to unit load acting along the girder and the arch respectively from the first and second step, the consistent deformation relations can then be applied for each node point. Since the consistent deformation relationships require that the deflection of the girder at each node point equal to the deformation of the vertical member plus the deflection of the two-hinged arch at the same node point, these relationships can be set up for each node point as a function of the vertical member redundant forces. The vertical redundant forces can then be obtained by solving these simultaneous equations. With the redundant vertical member forces known, it is possible to calculate the redundant horizontal reactions or the whole structure by superposition. Knowing all the redundant forces acting on the structure, the bending moment for each section of the girder and the arch can be calculated. The steps required in this analysis can be outlined as follows:

(1) Computation of the deflections of the simple girder.

$k = 1,2,\ldots,n.$
$j = 1,2,\ldots,n.$
$m = n+1.$

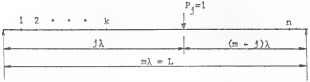


Fig. 2A. Notation for node points of a simple girder.

Subdividing the girder with node points and placing the loading, $P_j = 1$, at node point j as shown in Fig. 2A, the deflections of the beam at any section k can be found by using the conjugate beam method.

For the loading shown in Fig. 2A, the M/EI diagram is shown in Fig. 2B.

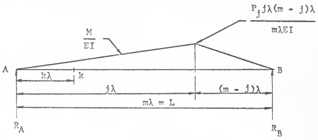


Fig. 2B. Loaded conjugate beam.

This M/EI diagram is the loading diagram for the conjugate beam as shown in Fig. 2B. The left hand reaction, R_A, of this conjugate beam can be found by taking moments about B:

$$R_A = \frac{1}{EIL} \; \frac{P_j(j\lambda)(m - j)\lambda}{m\lambda} \; \frac{j\lambda}{2} \left[\frac{j\lambda}{3} + (m - j)\lambda\right]$$

$$\frac{P_j(j\lambda)(m - j)\lambda}{m\lambda} \; \frac{(m - j)\lambda}{2} \; \frac{2(m - j)\lambda}{3}$$

$$= \frac{P_j j(m - j)\lambda^2}{6mEI} \; (2m - j) \quad .$$

Since the moment at a section in the conjugate beam is the deflection of the real beam at that section, the deflection at any section k of the real beam can be found by taking moments about node point k of the conjugate beam.

For $k \leq j$:

$$\delta_{kj} = R_A k\lambda - \frac{P_j(j\lambda)(m - j)\lambda}{m\lambda EI} \; \frac{k}{j} \; \frac{k\lambda}{2} \; \frac{k\lambda}{3}$$

$$= \frac{P_j j(m - j)(2m - j)\lambda^2}{6mEI} \, k\lambda - \frac{P_j(m - j)\lambda^3 k^3}{6mEI}$$

$$= \frac{L^3 k(m - j)\left[j(2m - j) - k^2\right]}{6m^4 EI} \quad . \qquad \text{(1-a)}$$

For $k \geq j$:

$$\delta_{kj} = \frac{L^3 j(m - k)\left[k(2m - k) - j^2\right]}{6m^4 EI} \quad . \qquad \text{(1-b)}$$

The first and second subscripts denote section and loading point respectively and are used throughout this report unless otherwise specified.

(2) Computation of the deflections of the two-hinged arch.

Using the method of virtual work, the vertical deflection of any section k, with loading, $P_j = 1$, acting at node point j, as shown in Fig. 3 can be

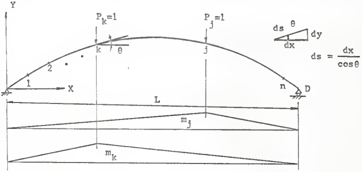


Fig. 3. Moment diagram of a simple parabolic arch with variable cross sections.

found by the following formula (2)(3).

$$\delta_{kj}^{sr} = \int_0^L \frac{m_j m_k dx}{EI^r \cos\theta} + \int_0^L \frac{n_j n_k dx}{A^r E \cos\theta} . \tag{2}$$

If the load P_k is acting horizontally at hinge point D, the corresponding values of moment m_k and thrust n_k are y and $\cos\theta$ respectively. Thus if y and $\cos\theta$ are substituted for m_k and n_k respectively in equation (2), the horizontal displacement of point D due to the vertical load, $P_j = 1$, can be found. Furthermore, if y and $\cos\theta$ are substituted for m_j and n_j respectively in equation (2), the value of δ_{kj}^{sr} is the horizontal displacement of point D due to a unit horizontal load acting at point D. If the thrust effects are neglected, equation (2) can be reduced to

$$\delta_{kj}^{sr} = \frac{1}{E} \int_0^L \frac{m_k m_j dx}{I^r \cos\theta} . \tag{3}$$

Knowing the deflections of a simple arch, the superposition method (Fig. 4) can now be used to find the deflections of a two-hinged arch.

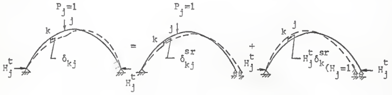


Fig. 4. Deflections of a two-hinged arch.

With the loading, $P_j = 1$, acting at node point j, the expression for the deflections at node point k can be expressed as follows (3):

$$\delta_{kj} = \delta_{kj}^{sr} - H_j^t \, \delta_{k(H_j=1)}^{sr} . \tag{4}$$

Where

$$H_j^t = \frac{\frac{1}{E}\int_0^S \frac{m_j y ds}{I^r} + \frac{1}{E}\int_0^S \frac{n_j \cos\theta ds}{A^r}}{\frac{1}{E}\int_0^S \frac{y^2 ds}{I^r} + \frac{1}{E}\int_0^S \frac{\cos^2\theta ds}{A^r}}$$

$$= \frac{\int_0^L \frac{m_j y dx}{I^r \cos\theta} + \int_0^L \frac{n_j dx}{A^r}}{\int_0^L \frac{y^2 dx}{I^r \cos\theta} + \int_0^L \frac{\cos\theta dx}{A^r}} . \tag{5}$$

It can be seen that the horizontal reaction H_j^t in equation (5) also comes from equation (4) by setting the left hand side of the equation equal to zero. This is simply the superposition method for calculating the horizontal reaction of a two-hinged arch. The numerator in the right hand side of equation (5) represents the horizontal displacement due to a unit vertical load, $P_j = 1$, acting at point j, and the denominator represents the horizontal displacement due to a unit load acting horizontally at point D. These interpretations can be made based on equation (2) as stated before.

If the influence of thrust in the arch is neglected, equation (5) can be reduced to

$$H_j^t = \frac{\int_0^L \frac{m_j y dx}{I^r \cos\theta}}{\int_0^L \frac{y^2 dx}{I^r \cos\theta}} \; . \tag{6}$$

(3) Calculation of the redundant forces in the vertical members.

Fig. 5 shows the whole system of a girder-arch bridge. For the loading, $P_j = 1$, acting at j, there are forces induced in every vertical member as shown. From the relations of consistent deformations, the deflection of the

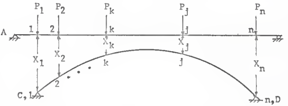


Fig. 5. Redundant forces in the ties of a girder-arch bridge.

girder at each node point must be equal to the deformation of the vertical member plus the deflection of the arch at the same node point. This relationship can be expressed as:

$$\delta_k^g = \delta_k^v + \delta_k^r \quad . \tag{7}$$

The deflections of the simple girder, δ_k^g, and of the two-hinged arch, δ_k^r, have already been calculated by the use of equations (1) and (4) respectively, and the deformation of the vertical member, δ_k^v, is given by:

$$\delta_k^v = \frac{X_k L_k}{A^v E} \quad .$$

With the loads acting on the structure as shown in Fig. 5, it may seen that the various deflection at point 1 can be expressed as:

$$\delta_1^g = \delta_{1,1}^g(P_1 - X_1) + \delta_{1,2}^g(P_2 - X_2) + \cdots + \delta_{1,n}^g(P_n - X_n) \; ,$$

$$\delta_1^v = \frac{X_1 L_1}{A^v E} \; ,$$

$$\delta_1^r = 0 \quad .$$

Substituting these values into equation (7), it follows that:

$$\delta_{1,1}^g(P_1 - X_1) + \delta_{1,2}^g(P_2 - X_2) + \cdots + \delta_{1,n}^g(P_n - X_n) = \frac{X_1 L_1}{A^v E} \quad . \tag{a}$$

Similarly, the deflections at point 2 can be expressed as:

$$\delta_2^g = \delta_{2,1}^g(P_1 - X_1) + \delta_{2,2}^g(P_2 - X_2) + \cdots + \delta_{2,n}^g(P_n - X_n) \; ,$$

$$\delta_2^v = \frac{X_2 L_2}{A^v E} \; ,$$

$$\delta_2^r = 0 + \delta_{2,2}^r X_2 + \delta_{2,3}^r X_3 + \cdots + \delta_{2,n-1}^r X_{n-1} + 0 \quad .$$

Substituting these values into equation (7), it follows that:

$$\delta_{2,1}^g (P_1 - X_1) + \delta_{2,2}^g (P_2 - X_2) + \cdots + \delta_{2,n}^g (P_n - X_n)$$

$$= \frac{X_2 L_2}{A^v E} + \delta_{2,2}^r X_2 + \delta_{2,3}^r X_3 + \cdots + \delta_{2,n-1}^r X_{n-1} \tag{b}$$

Similar expressions can be derived to express the deflection relationships for the other node points. Rearranging equations (a),(b),..., it follows that:

$$(\delta_{1,1}^g + \frac{L_1}{A^v E}) X_1 + \delta_{1,2}^g X_2 + \cdots + \delta_{1,n}^g X_n$$

$$= \delta_{1,1}^g P_1 + \delta_{1,2}^g P_2 + \cdots + \delta_{1,n}^g P_n \; , \tag{a)'$$

and

$$\delta_{2,1}^g X_1 + (\delta_{2,2}^g + \delta_{2,2}^r + \frac{L_2}{A^v E}) X_2 + \cdots + (\delta_{2,n-1}^g + \delta_{2,n-1}^r) X_{n-1} + \delta_{2,n}^g X_n$$

$$= \delta_{2,1}^g P_1 + \delta_{2,2}^g P_2 + \cdots + \delta_{2,n}^g P_n \; , \tag{b)'$$

...

...

Expressing equations (a)', (b)', ... in matrix form, it follows that:

$$
\begin{bmatrix}
(\delta^g_{1,1}+\frac{L_1}{A^vE}) & \delta^g_{1,2} & \delta^g_{1,3} & \cdots & \delta^g_{1,n-1} & \delta^g_{1,n} \\
\delta^g_{2,1} & (\delta^g_{2,2}+\delta^r_{2,2}+\frac{L_2}{A^vE}) & (\delta^g_{2,3}+\delta^r_{2,3}) & \cdots & (\delta^g_{2,n-1}+\delta^r_{2,n-1}) & \delta_{2,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\delta^g_{n-1,1} & (\delta^g_{n-1,2}+\delta^r_{n-1,2}) & (\delta^g_{n-1,3}+\delta^r_{n-1,3}) & \cdots & (\delta^g_{n-1,n-1}+\delta^r_{n-1,n-1}+\frac{L_{n-1}}{A^vE}) & \delta_{n-1,n} \\
\delta^g_{n,1} & \delta^g_{n,2} & \delta^g_{n,3} & \cdots & \delta^g_{n,n-1} & (\delta^g_{n,n}+\frac{L_n}{A^vE})
\end{bmatrix}
\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} =
$$

$$\begin{bmatrix} \delta^g_{1,1} & \delta^g_{1,2} & \cdots & \delta^g_{1,n-1} & \delta^g_{1,n} \\ \delta^g_{2,1} & \delta^g_{2,2} & \cdots & \delta^g_{2,n-1} & \delta^g_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^g_{n-1,1} & \delta^g_{n-1,2} & \cdots & \delta^g_{n-1,n-1} & \delta^g_{n-1,n} \\ \delta^g_{n,1} & \delta^g_{n,2} & \cdots & \delta^g_{n,n-1} & \delta^g_{n,n} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \\ P_n \end{bmatrix} \tag{8}$$

or symbolically

$$[A][X] = [B][P] \quad , \tag{9}$$

and

$$[X] = [A]^{-1}[B][P] \quad . \tag{10}$$

For finding the influence lines for the girder and the arch, the $[P]$ matrix is taken as a unit matrix.

(4) Calculation of redundant horizontal reactions.

It may be seen from Fig. 4 that whenever a load acts at j, there is a corresponding horizontal reaction H^t_j, which has been found from equation (5). From Fig. 5, it may be also seen that whenever a system of loads acts on the structure, there is a corresponding system of vertical member forces $X_1, X_2, \ldots, X_n$ induced in the vertical members. The horizontal reactions corresponding to such "sets" of vertical redundant forces as calculated in the preceding section can be expressed as:

$$H^f_j = X_{2j}H^t_2 + X_{3j}H^t_3 + \cdots + X_{n-1,j}H^t_{n-1} \quad . \tag{11-a}$$

With the unit loading acting along the girder, equation (11-a) can be expressed in matrix form as follows:

$$\begin{bmatrix} H^f_1 \\ H^f_2 \\ \cdot \\ \cdot \\ \cdot \\ H^f_{n-1} \\ H^f_n \end{bmatrix} = \begin{bmatrix} X_{2,1} & X_{3,1} & \cdots & X_{n-2,1} & X_{n-1,1} \\ X_{2,2} & X_{3,2} & \cdots & X_{n-2,2} & X_{n-1,2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_{2,n-1} & X_{3,n-1} & \cdots & X_{n-2,n-1} & X_{n-1,n-1} \\ X_{2,n} & X_{3,n} & \cdots & X_{n-2,n} & X_{n-1,n} \end{bmatrix} \begin{bmatrix} H^t_2 \\ H^t_3 \\ \cdot \\ \cdot \\ \cdot \\ H^t_{n-2} \\ H^t_{n-1} \end{bmatrix} , \tag{11-b}$$

or symbolically

$$[H^f] = [X'][H^t] \quad . \tag{11-c}$$

It may be seen that the matrix $[X']$ is the transpose of matrix $[X]$ obtained from the preceding section if the first and the last rows are eliminated.

(5) Calculation of moment influence lines for the girder.

Knowing the redundant forces induced in the vertical members, it is readily seen from Fig. 5 that the moments in the girder can be expressed as:

$$\begin{aligned}
M^g_{i,1} &= m^g_{i,1}(1 - X_{1,1}) - m^g_{i,2}X_{2,1} - m^g_{i,3}X_{3,1} - \cdots - m^g_{i,n}X_{n,1} \quad , \\
M^g_{i,2} &= - m^g_{i,1}X_{1,2} + m^g_{i,2}(1 - X_{2,2}) - m^g_{i,3}X_{3,2} - \cdots - m^g_{i,n}X_{n,2} \quad , \\
&\cdots \\
M^g_{i,n} &= - m^g_{i,1}X_{1,n} - m^g_{i,2}X_{2,n} - m^g_{i,3}X_{3,n} - \cdots + m^g_{i,n}(1 - X_{n,n}) \quad .
\end{aligned} \tag{12-a}$$

If point i is allowed to vary, equation (12-a) can be expressed in matrix form as:

$$\begin{bmatrix}
M^g_{1,1} & M^g_{1,2} & \cdots & M^g_{1,n-1} & M^g_{1,n} \\
M^g_{2,1} & M^g_{2,2} & \cdots & M^g_{2,n-1} & M^g_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
M^g_{n-1,1} & M^g_{n-1,2} & \cdots & M^g_{n-1,n-1} & M^g_{n-1,n} \\
M^g_{n,1} & M^g_{n,2} & \cdots & M^g_{n,n-1} & M^g_{n,n}
\end{bmatrix} =$$

$$\begin{bmatrix} m^g_{1,1} & m^g_{1,2} & \cdots & m^g_{1,n-1} & m^g_{1,n} \\ m^g_{2,1} & m^g_{2,2} & \cdots & m^g_{2,n-1} & m^g_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m^g_{n-1,1} & m^g_{n-1,2} & \cdots & m^g_{n-1,n-1} & m^g_{n-1,n} \\ m^g_{n,1} & m^g_{n,2} & \cdots & m^g_{n,n-1} & m^g_{n,n} \end{bmatrix} \begin{bmatrix} 1-X_{1,1} & -X_{1,2} & \cdots & -X_{1,n-1} & -X_{1,n} \\ -X_{2,1} & 1-X_{2,2} & \cdots & -X_{2,n-1} & -X_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -X_{n-1,1} & -X_{n-1,2} & \cdots & 1-X_{n-1,n-1} & -X_{n-1,n} \\ -X_{n,1} & -X_{n,2} & \cdots & -X_{n,n-1} & 1-X_{n,n} \end{bmatrix}, \tag{12-b}$$

or symbolically

$$[M^g] = [m^g][Q^g] \; . \tag{12-c}$$

(6) Calculation of moment influence lines for the arch.

Using a line of reasoning similar to that explained in the preceding section, it can be seen that the final moment influence line for the arch can be expressed as:

$$\begin{bmatrix} M^r_{1,1} & M^r_{1,2} & \cdots & M^r_{1,n-1} & M^r_{1,n} \\ M^r_{2,1} & M^r_{2,2} & \cdots & M^r_{2,n-1} & M^r_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M^r_{n-1,1} & M^r_{n-1,2} & \cdots & M^r_{n-1,n-1} & M^r_{n-1,n} \\ M^r_{n,1} & M^r_{n,2} & \cdots & M^r_{n,n-1} & M^r_{n,n} \end{bmatrix} =$$

$$\begin{bmatrix} m^r_{1,1} & m^r_{1,2} & \cdots & m^r_{1,n-1} & m^r_{1,n} \\ m^r_{2,1} & m^r_{2,2} & \cdots & m^r_{2,n-1} & m^r_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m^r_{n-1,1} & m^r_{n-1,2} & \cdots & m^r_{n-1,n-1} & m^r_{n-1,n} \\ m^r_{n,1} & m^r_{n,2} & \cdots & m^r_{n,n-1} & m^r_{n,n} \end{bmatrix} \begin{bmatrix} X_{1,1} & X_{1,2} & \cdots & X_{1,n-1} & X_{1,n} \\ X_{2,1} & X_{2,2} & \cdots & X_{2,n-1} & X_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{n-1,1} & X_{n-1,2} & \cdots & X_{n-1,n-1} & X_{n-1,n} \\ X_{n,1} & X_{n,2} & \cdots & X_{n,n-1} & X_{n,n} \end{bmatrix}, \tag{13-a}$$

or symbolically

$$[M^r] = [m^r][X] . \tag{13-b}$$

b. Solution 2.

Considering the flexural stress effects only.

This solution, which uses the concept of consistent deformations but neglects the axial stress effects, is quite similar to solution 1 which considered both the flexural and axial stress effects. With the axial stress in the vertical members neglected, it may be seen that the girder can not deflect at points 1 and n, therefore the girder is now considered as a continuous girder with supports at A, 1, n and B as shown in Fig. 6.

To apply the consistent deformations relation at each node point, it is first necessary to calculate the deflections of the continuous girder for each node point. To calculate the deflections of the continuous girder for each node point, it is necessary to know the redundant reactions at supports 1 and n. This is accomplished by using the superposition method, taking advantage of the known values of the deflections of a simple girder as computed in solution 1. Knowing the values of the redundant reactions, the superposition method is used again to obtain the deflections of the continuous girder.

The second step is to find the deflections of a two-hinged arch. Since the axial stress effects are neglected, equations (3) and (6) are used instead of equation (2) and (5). Substituting the values from these two equations into equation (4), the deflections of a two-hinged arch can be calculated as stated previously in solution 1. Knowing the deflections of the continuous girder and of the two-hinged arch for each mode point, the relationship of consistent deformations can be applied to each node point, and the equations for the unknown vertical member forces can be set up. Solving the equations simultaneously and calculating the redundant horizontal reactions as done in solution 1, the redundants in the structure can be completely solved and the moment influence lines can be drawn. The steps involved in this analysis can be outlined as follows:

(1) Calculation of the deflections of the continuous girder.

Since the axial stress effects are neglected, the girder is considered as a continuous girder with supports at A, 1, n and B as shown in Fig. 6.

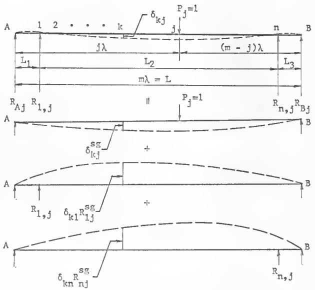


Fig. 6. Superposition of the deflections of the continuous girder.

Using the deflections of the simple girder as calculated from equations (1-a) and (1-b) in solution 1, the deflections of the continuous girder can be found by using the superposition method as shown in Fig. 6. Since the deflections of the girder at points 1 and n have to be zero, it follows that:

$$\delta_{1,1}R_{1,j} + \delta_{1,n}R_{n,j} - \delta_{1,j} = 0 ,$$

$$\delta_{n,1}R_{1,j} + \delta_{n,n}R_{n,j} - \delta_{n,j} = 0 .$$

Solving these two equations simultaneously, it follows that:

$$R_{1,j} = \frac{\delta_{1,j}\delta_{n,n} - \delta_{n,j}\delta_{1,n}}{\delta_{1,1}\delta_{n,n} - \delta_{1,n}\delta_{n,1}} ,$$

$$R_{n,j} \quad \frac{\delta_{n,j}\delta_{1,1} - \delta_{1,j}\delta_{n,1}}{\delta_{1,1}\delta_{n,n} - \delta_{1,n}\delta_{n,1}} . \tag{14}$$

Knowing the values of $R_{1,j}$ and $R_{n,j}$, the deflections of the continuous girder are:

$$\delta_{k,j} = \delta_{k,j}^{sg} - R_{1,j}\delta_{k,1}^{sg} - R_{n,j}\delta_{k,n}^{sg} . \tag{15}$$

(2) Calculation of the deflection of the two-hinged arch.

This step is exactly the same as stated in step (2) of solution 1. Equations (3), (4), and (6) are used in this solution whereas equations (2), (4), and (5) are used in solution 1.

(3) Calculation of the redundant forces in the vertical members.

This step can be seen to be similar to step (3) in solution 1. With the axial stress effects ignored, the term δ_k^v in equation (7) is omitted. Noting that in this solution there are no deflections at points 1 and n, it may be seen that the first and the last rows of the first matrix in the left hand side of equation (8) are omitted. Furthermore, noting that the redundants X_1 and X_n which are the values of $R_{1,j}$ and $R_{n,j}$ have been calculated, equation (8) can be expressed, in this solution, as:

$$\begin{bmatrix} \delta^g_{2,2}+\delta^r_{2,2} & \delta^g_{2,3}+\delta^r_{2,3} & \cdots & \delta^g_{2,n-2}+\delta^r_{2,n-2} & \delta^g_{2,n-1}+\delta^r_{2,n-1} \\ \delta^g_{3,2}+\delta^r_{3,2} & \delta^g_{3,3}+\delta^r_{3,3} & \cdots & \delta^g_{3,n-2}+\delta^r_{3,n-2} & \delta^g_{3,n-1}+\delta^r_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^g_{n-2,2}+\delta^r_{n-2,2} & \delta^g_{n-2,3}+\delta^r_{n-2,3} & \cdots & \delta^g_{n-2,n-2}+\delta^r_{n-2,n-2} & \delta^g_{n-2,n-1}+\delta^r_{n-2,n-1} \\ \delta^g_{n-1,2}+\delta^r_{n-1,2} & \delta^g_{n-1,3}+\delta^r_{n-1,3} & \cdots & \delta^g_{n-1,n-2}+\delta^r_{n-1,n-2} & \delta^g_{n-1,n-1}+\delta^r_{n-1,n-1} \end{bmatrix} \begin{bmatrix} X_2 \\ X_3 \\ \vdots \\ X_{n-2} \\ X_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \delta^g_{2,2} & \delta^g_{2,3} & \cdots & \delta^g_{2,n-2} & \delta^g_{2,n-1} \\ \delta^g_{3,2} & \delta^g_{3,3} & \cdots & \delta^g_{3,n-2} & \delta^g_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^g_{n-2,2} & \delta^g_{n-2,3} & \cdots & \delta^g_{n-2,n-2} & \delta^g_{n-2,n-1} \\ \delta^g_{n-1,2} & \delta^g_{n-1,3} & \cdots & \delta^g_{n-1,n-2} & \delta^g_{n-1,n-1} \end{bmatrix} \begin{bmatrix} P_2 \\ P_3 \\ \vdots \\ P_{n-2} \\ P_{n-1} \end{bmatrix}, \tag{16}$$

or expressed symbolically in the same form as equation (9):

$$[A][X] = [B][P] , \tag{17}$$

and

$$[X] = [A]^{-1}[B][P] . \tag{18}$$

(4) Calculation of the redundant horizontal reactions.

With a line of reasoning similar to that explained in step (4) of solution 1, and noting that whenever the loads act at points 1 and n they will cause no stress in the remaining vertical truss members, it may be seen that the first and the last rows in the matrices $[H^f]$ and $[X']$ of equation (11-c) are eliminated. Thus equation (11-b) can be expressed, in this solution, as:

$$\begin{bmatrix} H_2^f \\ H_3^f \\ \cdot \\ \cdot \\ \cdot \\ H_{n-2}^f \\ H_{n-1}^f \end{bmatrix} = \begin{bmatrix} X_{2,2} & X_{3,2} & \cdots & X_{n-2,2} & X_{n-1,2} \\ X_{2,3} & X_{3,3} & \cdots & X_{n-2,3} & X_{n-1,3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_{2,n-2} & X_{3,n-2} & \cdots & X_{n-2,n-2} & X_{n-1,n-2} \\ X_{2,n-1} & X_{3,n-1} & \cdots & X_{n-2,n-1} & X_{n-1,n-1} \end{bmatrix} \begin{bmatrix} H_2^t \\ H_3^t \\ \cdot \\ \cdot \\ \cdot \\ H_{n-2}^t \\ H_{n-1}^t \end{bmatrix} , \tag{19}$$

or symbolically,

$$[H^f] = [X'][H^t] . \tag{20}$$

It may be seen that the matrix $[X']$ is the transpose of the matrix $[X]$ obtained from the preceding section. Therefore equation (20) can be expressed as:

$$[H^f] = [X]^T[H^t] . \tag{21}$$

(5) Calculation of moment influence lines for the girder.

In order to draw the moment influence lines for the girder, the influence values of the moments induced in the continuous girder for each section due to a unit load acting along the girder must be known. With the values $R_{1,j}$ and $R_{n,j}$ from equation (14), the reactions at the end points can be found by taking moments with respect to either end of the girder

(Fig. 6) and can be expressed as:

$$R_{A,j} = \left[(m - j)\lambda - R_{1j}(L_2 + L_3) - R_{n,j}L_2\right]/L \quad ,$$
$$R_{B,j} = \left[j\lambda - R_{1,j}L_1 - R_{n,j}(L_1 + L_2)\right]/L \tag{22}$$

Knowing the values of $R_{A,j}$ and $R_{B,j}$, the moments at each section with the loading, $P_j = 1$, acting at j can be expressed as:

for $i \leq j$:

$$m_{i,j} = R_{A,j}(i\lambda) + R_{1,j}(i - 1)\lambda \quad , \tag{23-a}$$

for $i \geq j$:

$$m_{i,j} = R_{B,j}(m - i)\lambda + R_{n,j}(n - i)\lambda \quad . \tag{23-b}$$

Knowing all the values of $m_{i,j}$, the same arguments used in solution 1 for finding the moment influence lines can be applied in this solution. Noting that when the loads act at points 1 and n they will cause no moment at any section of the structure, the first and the last columns of the matrix $[M^g]$ and m^g in equation (12-c) are eliminated. Equation (12-b) can then be expressed, in this solution, as:

$$\begin{bmatrix} M^g_{1,2} & M^g_{1,3} & \cdots & M^g_{1,n-2} & M^g_{1,n-1} \\ M^g_{2,2} & M^g_{2,3} & \cdots & M^g_{2,n-2} & M^g_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M^g_{n-1,2} & M^g_{n-1,3} & \cdots & M^g_{n-1,n-2} & M^g_{n-1,n-1} \\ M^g_{n,2} & M^g_{n,3} & \cdots & M^g_{n,n-2} & M^g_{n,n-1} \end{bmatrix} =$$

$$\begin{bmatrix} m^g_{1,2} & m^g_{1,3} & \cdots & m^g_{1,n-2} & m^g_{1,n-1} \\ m^g_{2,2} & m^g_{2,3} & \cdots & m^g_{2,n-2} & m^g_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m^g_{n-1,2} & m^g_{n-1,3} & \cdots & m^g_{n-1,n-2} & m^g_{n-1,n-1} \\ m^g_{n,2} & m^g_{n,3} & \cdots & m^g_{n,n-2} & m^g_{n,n-1} \end{bmatrix} \begin{bmatrix} 1-X_{2,2} & -X_{2,3} & \cdots & -X_{2,n-2} & -X_{2,n-1} \\ -X_{3,2} & 1-X_{3,3} & \cdots & -X_{3,n-2} & -X_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -X_{n-2,2} & -X_{n-2,3} & \cdots & 1-X_{n-2,n-2} & -X_{n-2,n-1} \\ -X_{n-1,2} & -X_{n-1,3} & \cdots & -X_{n-1,n-2} & 1-X_{n-1,n-1} \end{bmatrix}, \tag{24}$$

or symbolically,

$$[M^g] = [m^g][Q^g] \; . \tag{25}$$

(6) Calculation of moment influence lines for the arch.

With the same interpretations as those stated in step (6) of solution 1, and noting that there are no deflections or moments when the loading acts at points 1 and n, equation (13-a) can be expressed, in this solution, as:

$$\begin{bmatrix} M^r_{2,2} & M^r_{2,3} & \cdots & M^r_{2,n-2} & M^r_{2,n-1} \\ M^r_{3,2} & M^r_{3,3} & \cdots & M^r_{3,n-2} & M^r_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M^r_{n-2,2} & M^r_{n-2,3} & \cdots & M^r_{n-2,n-2} & M^r_{n-2,n-1} \\ M^r_{n-1,2} & M^r_{n-1,3} & \cdots & M^r_{n-1,n-2} & M^r_{n-1,n-1} \end{bmatrix} =$$

$$\begin{bmatrix} m^r_{2,2} & m^r_{2,3} & \cdots & m^r_{2,n-2} & m^r_{2,n-1} \\ m^r_{3,2} & m^r_{3,3} & \cdots & m^r_{3,n-2} & m^r_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m^r_{n-2,2} & m^r_{n-2,3} & \cdots & m^r_{n-2,n-2} & m^r_{n-2,n-1} \\ m^r_{n-1,2} & m^r_{n-1,3} & \cdots & m^r_{n-1,n-2} & m^r_{n-1,n-1} \end{bmatrix} \begin{bmatrix} X_{2,2} & X_{2,3} & \cdots & X_{2,n-2} & X_{2,n-1} \\ X_{3,2} & X_{3,3} & \cdots & X_{3,n-2} & X_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{n-2,2} & X_{n-2,3} & \cdots & X_{n-2,n-2} & X_{n-2,n-1} \\ X_{n-1,2} & X_{n-1,3} & \cdots & X_{n-1,n-2} & X_{n-1,n-1} \end{bmatrix}, \tag{26}$$

or symbolically,

$$[M^r] = [m^r][X] \; . \tag{27}$$

II. Energy method.

Since the structure analysed in this report is 14 degrees statically indeterminate, the ordinary energy method which requires writing the energy terms for each segment of the structure seems too tedious. The problem is treated here by using a method suggested by Professor James Michalos (4). Using this method, the portion 1,C,D,n of the structure shown in Fig. 7 is considered as a compound section with flexural rigidity, EI, equal to the sum of that of the girder and of the arch in the X-axis direction. To find the moments in the girder and in the arch, it is necessary to find the reaction redundants and to calculate the moments induced in each compound section. Then using equation (39), which will be discussed later, the total moment is distributed between the girder and the arch. The moment influence lines can then be drawn. The steps required in this analysis can be outlined as follows:

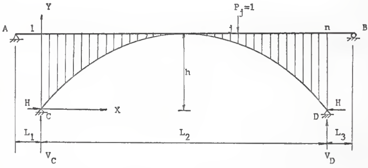


Fig. 7. The reduced structure of a girder-arch bridge.

(1) Calculation of the redundant reaction influence lines for the girder-arch bridge.

The method of virtual displacements (5) is used to find the influence lines for the reactions V_c, V_D and H as shown in Fig. 7. Since the theory of virtual displacements states that the deflection curve of a structure due to unit displacement in the direction of the unknown force represents the influence line of that unknown force, the influence lines for V_c, V_D and H can be found as follows:

The moments caused by the three redundant reactions V_c, V_D and H are shown in Fig. 8 (b), (c), and (d) respectively. The total moment induced in the structure by these redundant reactions is the sum of the three moments. Knowing the total moment, which is a function of V_c, V_D and H, for each section, it may be seen that the total internal strain energy U can be expressed as:

$$U = \int_0^{L_1} \frac{(m_c + m_d)^2 dx}{2EI^g} + \int_{L_1}^{L_1+L_2} \frac{(m_c + m_d + m_h)^2 dx}{2E(I^g + I^r\cos\theta)} + \int_{L_1+L_2}^{L} \frac{(m_c + m_d)^2 dx}{2EI^g}$$

$$= \int_0^{L_1} \frac{(V_C m_1 + V_D m_n)^2 dx}{2EI^g} + \int_{L_1}^{L_1+L_2} \frac{(V_C m_1 + V_D m_n + Hy)^2 dx}{2E(I^g + I^r\cos\theta)}$$

$$+ \int_{L_1+L_2}^{L} \frac{(V_C m_1 + V_D m_n)^2 dx}{2EI^g} \quad . \qquad (28)$$

According to Castingliano's first theorem (2), the first partial derivative of the total strain energy of the structure with respect to the applied action denotes the deflection component of the point of application of an action on a structure, in the direction of that action. Or expressed mathematically:

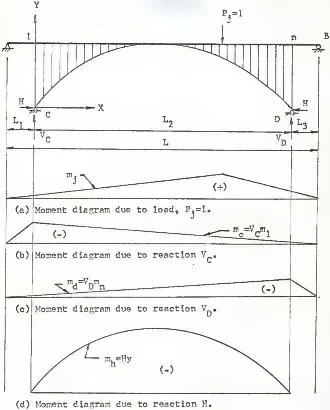


Fig. 8. Moment diagrams due to external loading and redundant reactions.

$$\frac{\partial U}{\partial V_C} = \Delta_C \, ,$$

$$\frac{\partial U}{\partial V_D} = \Delta_D \, , \tag{29}$$

$$\frac{\partial U}{\partial H} = \Delta_H \, .$$

Substituting equation (28) into equation (29), it follows that:

$$\Delta_H = \frac{\partial U}{\partial H} = \frac{V_C}{E}\left[\int_{L_1}^{L_1+L_2} \frac{y m_1 dx}{(I^g + I^r \cos\theta)}\right] + \frac{V_D}{E}\left[\int_{L_1}^{L_1+L_2} \frac{y m_n dx}{(I^g + I^r \cos\theta)}\right]$$

$$+ \frac{H}{E}\left[\int_{L_1}^{L_1+L_2} \frac{y^2 dx}{(I^g + I^r \cos\theta)}\right] \quad \text{(30-a)}$$

$$\Delta_C = \frac{\partial U}{\partial V_C} = \frac{V_C}{E}\left[\int_0^{L_1} \frac{(m_1)^2 dx}{I^g} + \int_{L_1}^{L_1+L_2} \frac{(m_1)^2 dx}{(I^g + I^r \cos\theta)} + \int_{L_1+L_2}^{L} \frac{(m_1)^2 dx}{I^g}\right]$$

$$+ \frac{V_D}{E}\left[\int_0^{L_1} \frac{m_1 m_n dx}{I^g} + \int_{L_1}^{L_1+L_2} \frac{m_1 m_n dx}{(I^g + I^r \cos\theta)} + \int_{L_1+L_2}^{L} \frac{m_1 m_n dx}{I^g}\right]$$

$$+ \frac{H}{E}\left[\int_{L_1}^{L_1+L_2} \frac{y m_1 dx}{(I^g + I^r \cos\theta)}\right] \quad \text{(30-b)}$$

$$\Delta_D = \frac{\partial U}{\partial V_D} = \frac{V_C}{E}\left[\int_0^{L_1} \frac{m_1 m_n dx}{I^g} + \int_{L_1}^{L_1+L_2} \frac{m_1 m_n dx}{(I^g + I^r \cos\theta)} + \int_{L_1+L_2}^{L} \frac{m_1 m_n dx}{I^g}\right]$$

$$+ \frac{V_D}{E}\left[\int_0^{L_1} \frac{(m_n)^2 dx}{I^g} + \int_{L_1}^{L_1+L_2} \frac{(m_n)^2 dx}{(I^g + I^r \cos\theta)} + \int_{L_1+L_2}^{L} \frac{(m_n)^2 dx}{I^g}\right]$$

$$+ \frac{H}{E}\left[\int_{L_1}^{L_1+L_2} \frac{y m_n dx}{(I^g + I^r \cos\theta)}\right] \quad \text{(30-c)},$$

or symbolically,

$$E\Delta_H = E\frac{\partial U}{\partial H} = aV_C + bV_D + cH$$

$$E\Delta_C = E\frac{\partial U}{\partial V_C} = dV_C + eV_D + fH \tag{31}$$

$$E\Delta_D = E\frac{\partial U}{\partial V_D} = gV_C + hV_D + iH .$$

or in matrix form,

$$E\begin{bmatrix}\Delta_H \\ \Delta_C \\ \Delta_D\end{bmatrix} = E\begin{bmatrix}\frac{\partial U}{\partial H} \\ \frac{\partial U}{\partial V_C} \\ \frac{\partial U}{\partial V_D}\end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i\end{bmatrix}\begin{bmatrix}V_C \\ V_D \\ H\end{bmatrix} = [A]\begin{bmatrix}V_C \\ V_D \\ H\end{bmatrix}. \tag{32}$$

Using the theory of virtual displacements, the influence values of V_C, V_D and H, which are used to draw the influence lines, can be found by setting Δ_H, Δ_C and Δ_D, in turn, equal to unity while the remaining terms in the same set of displacements is held equal to zero, i.e.

$$E\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix} = [A]\begin{bmatrix}V_C^1 & V_C^2 & V_C^3 \\ V_D^1 & V_D^2 & V_D^3 \\ H^1 & H^2 & H^3\end{bmatrix}. \tag{33}$$

The subscripts 1,2 and 3 represent the order of sets of the values of V_C, V_D and H, the total moment, M_{kt}^i, can be expressed as:

$$M_{kt}^i = V_C^i m_{kl} + V_D^i m_{kn} + H^i y_k , \qquad i=1,2,3. \tag{34}$$

The deflections of the structure can then be found by using the method of virtual work and can be expressed as:

$$\delta_k^i = \int_0^{L_1} \frac{M_{kt}^i m_k dx}{EI^g} + \int_{L_1}^{L_1+L_2} \frac{M_{kt}^i m_k dx}{E(I^g + I^r \cos\theta)} + \int_{L_1+L_2}^{L} \frac{M_{kt}^i m_k dx}{EI^g} , \qquad i=1,2,3. \tag{35}$$

Equation (35) gives three sets of values of deflections which are the influence lines for H, V_C and V_D respectively.

(2) Calculation of moment influence lines for the girder and the arch.

Knowing the influence lines for V_C, V_D and H, the total moment induced in each section of the structure when the structure is loaded can be found by adding the four moment diagrams as shown in Fig. 8. The total moment, M_t, can be distributed between the girder and the arch by the following method (4).

The structure will deform when the loads act on it. If the angle changes along the girder and the arch are equal at each corresponding point, that is, if:

$$\frac{M^r dx}{EI^r \cos\theta} = \frac{M^g dx}{EI^g} , \tag{36}$$

where θ is the angle between the X-axis and tangent of the rib at each node point. Then

$$\frac{M^r}{M^g} = \frac{I^r \cos\theta}{I^g} . \tag{37}$$

If we denote M_t as the total moment that is resisted by bending in the arch rib and girder at the section, that is,

$$M_t = M^g + M^r , \tag{38}$$

then from equation (37),

$$M^g = M_t\left(\frac{I^g}{I^g + I^r\cos\theta}\right) \quad ,$$

$$M^r = M_t\left(\frac{I^r\cos\theta}{I^g + I^r\cos\theta}\right) \quad . \tag{39}$$

Where M_t can be found by the following equation as stated previously:

$$M_t = m_{ij} - V_C m_{il} - V_D m_{in} - Hy_i \quad . \tag{40}$$

Equation (39) thus provides a simple method for drawing the moment influence lines for the girder and the arch.

NUMERICAL EXAMPLE

All of the calculations for this example are carried out using an IBM 1620 digital computer. Fig. 9 shows the dimensions and properties of the bridge analysed in this example. Three solutions of this numerical example were accomplished. The first and second solutions used the consistent deformations method with the first solution taking into account both the effects of flexural and of axial stresses and the second solution considering the flexural stresses only. The third solution was accomplished by using the energy method. All solutions are explained separately below.

$$\text{ARCH EQUATION :} \quad y = \frac{x}{375}(300 - x).$$

$$I^r_{cr} = 12 I^g \ (ft^4). \qquad A^v = I^r_{cr}/3 \ (ft^2).$$

$$I^r_j = I^r_{cr}\cos\theta_j \ (ft^4). \qquad A^r_j = I_j \ (ft^2).$$

$$E = 30 \times 10^6 \text{ psi}.$$

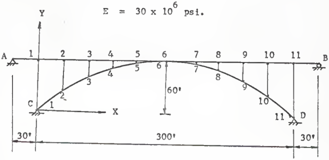


Fig. 9. Dimensions and properties of a girder-arch bridge.

Solution 1.

Considering both the flexural and axial stress effects and using the method of consistent deformations.

Solution:

Following the steps outlined in the General Analysis Procedures for solution 1, the final results are obtained and shown in Fig. 10 to Fig. 13. The appropriate data are shown in Tables 1 to 5 in Appendix A. By the theory of virtual displacements, it can be seen that the shape of each influence line is reasonable since they are generally of the shape of the deflection curve. This solution provides the most accurate results since it takes into account the most stress effects.

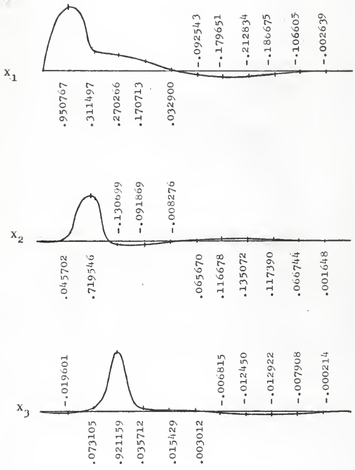
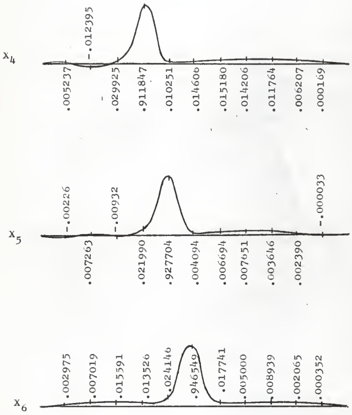


Fig.10. Influence lines for the redundant vertical member forces in solution 1.


x4
-.012395
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.029925
.911847
.010251
.014606
.015180
.014206
.011764
.006207
.000169
x5
-.00226
-.00932
-.000033
.007263
.021990
.927704
.004094
.006694
.007651
.003646
.002390
x6
.002975
.007019
.015591
.013526
.024146
.946549
.017741
.005000
.008939
.002065
.000352

Fig. 10. (Contd.)

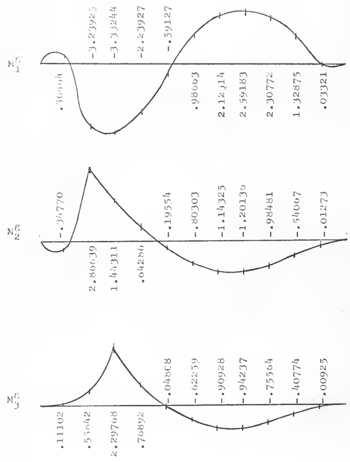
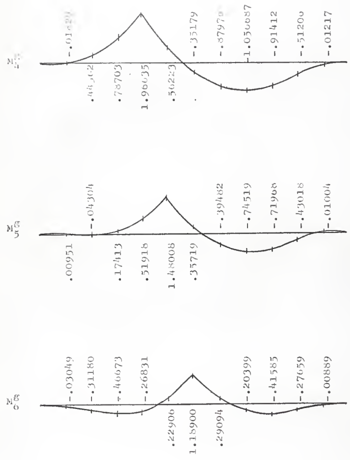


Fig. 11. Moment influence lines for the girder in solution 1.



-.33179
-1.05687
-.91412
-.01217
.78703
M5g
-.04304
-.39482
-.74519
-.71968
-.43018
-.01004
.00951
.17413
.51918
1.48008
.35719
M6g
-.03049
-.31180
-.26831
-.20399
-.41585
-.27659
-.00889
.22906
1.18900
.29094

Fig. 11. (Contd.)

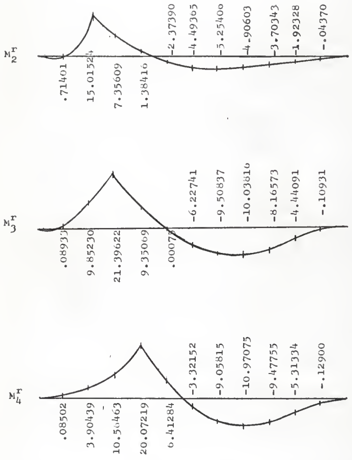
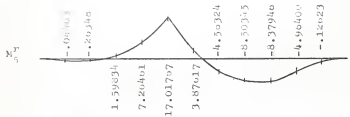


Fig.12. Moment influence lines for the arch in solution 1.



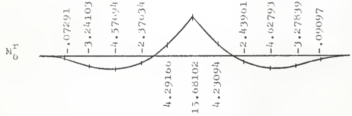


Fig.12. (Contd.)

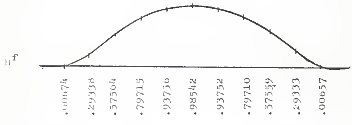


Fig.13. Final influence line for H^f in solution 1.

Solution 2.

Considering the flexural stress effects only and using the method of consistent deformations.

Solution:

Following the steps outlined in the General Analysis Procedures for solution 2, the final results are obtained and shown in Fig. 14 to Fig. 17. The appropriate data are shown in Table 1 to 5 in Appendix B. It can be seen that most of the results are quite close to those of solution 1. It is predictable that the results for the influence line for X_1(or R_1) would not be very accurate since the actual structural behaviour for this member, which is the longest among the vertical truss members, is quite far removed from the assumption, of negligible axial stress. It may be noted that because of the assumptions made and the node points selected there is no information on the moment influence lines for the girder and arch for the segments between node points A, 1 and 11, B.

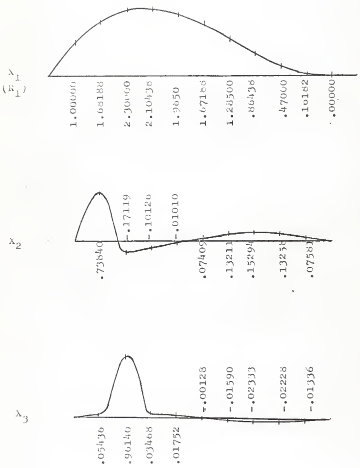


Fig.14. Influence lines for the redundant vertical member forces in solution 2.

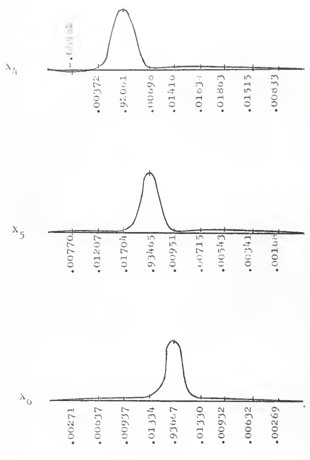

x_4
.00372
.92061
.00696
.01416
.01634
.01863
.01515
.00833
x_5
.00770
.01207
.01704
.93465
.00951
.00715
.00543
.00341
.00168
x_6
.00271
.00637
.00937
.01334
.93667
.01330
.00932
.00632
.00269

Fig.14. (Contd.)

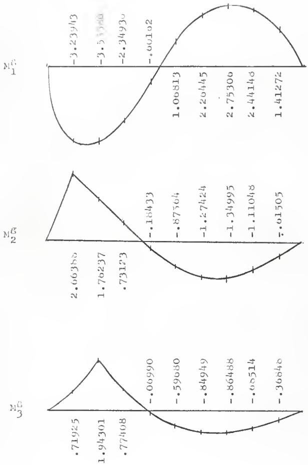
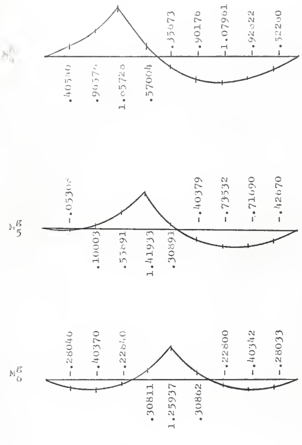


Fig.15. Moment influence lines for the girder in solution 2.



.40540
1.05728
.57004
-.35673
-.90176
-1.07961
.10003
1.41933
.30891
-.40379
-.73532
-.71690
-.42670
-.28046
-.40370
-.22840
.30811
1.25937
.30862
-.22800
-.40342
-.28033

Fig.15. (Contd.)

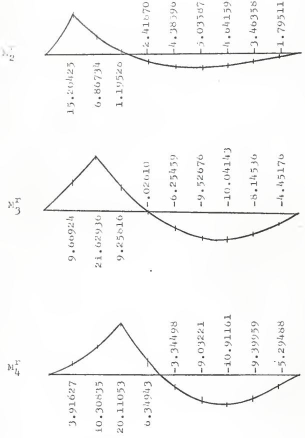


Fig.16. Moment influence lines for the arch in solution 2.

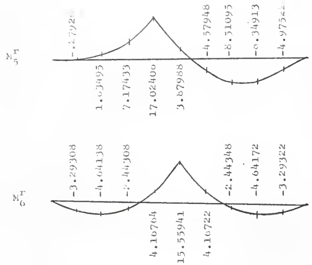


Fig.16. (Contd.)

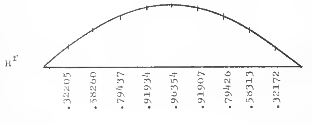


Fig.17. Influence line for H^f in Solution 2.

Solution 3. Energy method solution.

Solution:

Following the steps outlined in the General Analysis Procedures for solution 3, the final results are obtained and shown in Fig. 18, Fig. 19 and Fig. 20. The appropriate data are shown in Table 1 and Table 2 in Appendix C. It may be seen that the results are quite close to those of solutions 1 and 2. Since in this method of solution, the vertical member forces are not relevant there is no information obtained concerning them. Checking the results, it can be seen that the moments at points 1 and 11 are quite far removed from the values obtained from solutions 1 and 2. This can be explained by the fact that the actual structural behaviour differs a great deal from the assumed behavior. Points 1 and 11 are hinged for the arch while the corresponding points in the girder are continuous. Therefore the assumption that the curvature due to load in the arch is equal to the curvature due to load in the girder is in reality a relatively poor assumption.

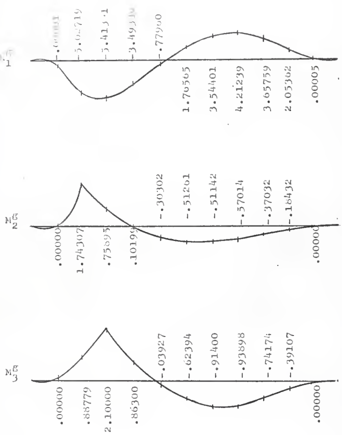


Fig.18. Moment influence lines for the girder in solution 3.

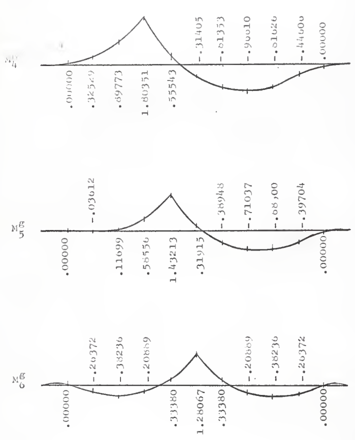

M4
-.31405
1.80351
.55543
.69773
.00000
Mg5
.00000
-.03612
.11699
1.43213
.31915
-.38948
-.71037
-.39704
.00000
Mg6
.00000
-.26372
-.38236
-.20889
.33380
1.28067
.33380
-.20889
-.38236
-.26372
.00000

Fig.18. (Contd.)

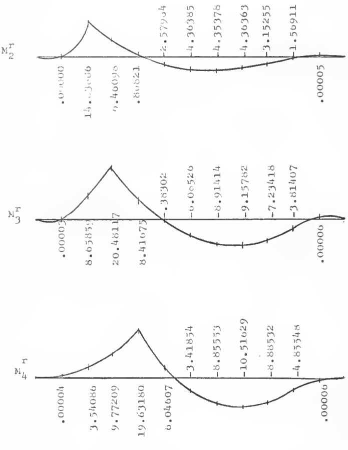
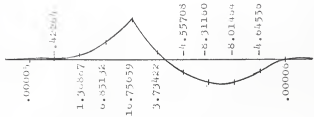


Fig.19. Moment influence lines for the arch in solution 3.

M_5^r



M_6^r

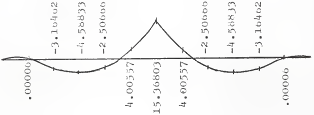


Fig.19. (Contd.)

H^f

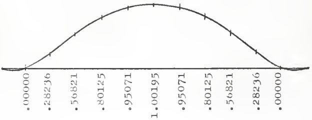


Fig.20. Final influence line for H^f in solution 3.

A comparison between the results of solution 1, 2, and 3 shows that they are reasonably close to each other. The first solution provides the most accurate results but requires the most computational effort. The second solution, which requires less computational effort, provides very accurate results in comparison with those of solution 1. The last solution provides the most inaccurate results. They are, however, still reasonably close and require considerably less computation than either of the other solutions. Of these three solutions the third solution might well be used for a preliminary design while either of the other two might be used for the final design.

CONCLUSIONS

By comparing the three sets of solutions of the numerical examples, it can be seen that the results correspond closely.

For solution 1 the concept of consistent deformations which is very familiar to structural engineers was used. With the consideration of thrust in the arch and the axial stresses in the vertical members, the procedures of computation are much more troublesome than they are in the other two solutions. However, this is the most accurate solution of the three. Solution 2 makes use of the same concept of consistent deformations but includes the assumption that the thrust and axial stress effects can be ignored. The results are nearly the same as those of the first solution. Hence it would seem that it is satisfactory to use this solution in a practical design. In the last solution, which makes use of the general concept of minimum energy and the theory of virtual displacements, the redundants are greatly reduced from 14 degrees to 3 degrees. This reduction of redundants is a great advantage in the computations and makes it practical to carry them out by hand computations in case a computer is unavilable. As a matter of fact, the energy method can be used in the preliminary design without resulting in significant error.

It should be noted that the moment influence lines for point 1 and 2 for the energy method are quite far removed from those obtained from solution 1 and solution 2 by the consistent deformations method. This discrepancy can be explained by the assumption made in the energy method solution that the angle changes along the girder and the arch rib are equal. It can be seen that actually the arch is hinged at point c which corresponds to point 1 of the girder which is continuous at that point. That is, the assumption of equal change of angles along the girder and the arch at the corresponding points of the structure deviates significantly from the actual structural behaviour. It is not therefore surprising that the various solutions of the problem also have significant deviations at these points. If the arch portion were a fixed-ended arch, the differences between the results should be greatly decreased.

ACKNOWLEDGEMENT

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NOTATION

A^r	:	Cross sectional area of arch rib.
A^v	:	Cross sectional area of vertical hinged members.
E	:	Modulus of elasticity.
H	:	Horizontal reaction.
H_j^f	:	Final horizontal reaction of the girder arch bridge due to loads acting at node point j.
H_j^t	:	Horizontal reaction of a single two-hinged arch due to loads acting at node point j of a two-hinged arch.
I_{cr}^r	:	Moment of inertia at the crown of arch.
I^g	:	Moment of inertia of girder.
I^r	:	Moment of inertia of arch.
L	:	Length measured in the X-axis direction.
L_j	:	Length of the vertical member j.
m_c, m_d, m_h	:	Moment caused by V_C, V_D, and H in the reduced structure in the energy method respectively.
m_j, m_k, etc.	:	Moment due to loads acting at node point j, k, etc.
m_{ij}^g, m_{ij}^r	:	Moment induced at node point i caused by a unit load acting at node point j for girder and arch respectively.
M	:	Moment.
M_{ij}^g, M_{ij}^r	:	Total inducod moment in girder and arch respectively.
M_{kt}	:	Total induced moment at section k of the compound section of girder and arch in the energy method.
n_j, n_k, etc.	:	Thrust due to loads acting at node points j, k, etc.
P_j, P_k, etc.	:	Loads acting at node points j, k, etc.
R_{ij}	:	Reaction at point i due to loads acting at point j.
S	:	Length measured along arch.

U	:	Strain energy.
V_C, V_D	:	Vertical reactions at hinged point C, D respectively.
X_i, X_j, etc.	:	Forces in the vertical truss members i, j, etc.
X_{ij}	:	Forces induced in vertical member i due to loads acting at node point j.
λ	:	Length of segments between adjacent node points measured in the X-axis direction.
δ_k^g, δ_j^g, etc.	:	Vertical deflections of girder at section k, j, etc.
δ_k^v, δ_j^v, etc.	:	Shortening or elongation of the vertical members k, j, etc.
δ_k^r, δ_j^r, etc.	:	Vertical deflection of arch at section k, j, etc.
δ_{kj}	:	Deflection at section k due to loading acting at node point j.
δ_{kj}^{sg}	:	Deflection at section k of a simple girder with loads acting at node point j.
δ_{kj}^{sr}	:	Deflection at section k of a simply supported arch with loads acting at node point j of the simply supported arch.
Δ_C, Δ_D	:	Vertical displacement of hinged points C and D respectively in the energy method.
Δ_H	:	Relative horizontal displacement of hinged points C and D in the energy method.
θ_j	:	Angle between the X-axis and the tangent of the arch at point j.

APPENDIX A

	δ1	δ2	δ3	δ4	δ5	δ6
1	[illegible]7[illegible].	[illegible].	[illegible].	[illegible].	[illegible]750.	[illegible]750.
2	[illegible].	[illegible].	[illegible].	[illegible].	427750.	[illegible].
3	[illegible].	[illegible].	[illegible].	[illegible].	[illegible].	[illegible].
4	[illegible].	[illegible].	[illegible].	[illegible].	[illegible].	[illegible].
5	[illegible].	[illegible].	[illegible].	[illegible].	[illegible]750.	[illegible]3375[illegible].
6	[illegible].	[illegible].	[illegible].	[illegible].	[illegible]750.	[illegible]72000.
7	[illegible].	[illegible].	[illegible].	[illegible].	[illegible]1250.	[illegible]33750.
8	[illegible].	[illegible].	[illegible].	[illegible].	772500.	828000.
9	[illegible].	[illegible].	[illegible].	[illegible].	61[illegible]750.	[illegible].
10	[illegible].	[illegible].	[illegible]750.	[illegible].	431250.	[illegible].
11	[illegible].	[illegible].	[illegible].	[illegible].	221250.	240750.

TABLE 1. DEFLECTIONS OF A SIMPLE [illegible].

	δ2	δ3	δ4	δ5	δ6
2	[illegible]242[illegible].	[illegible]265.	[illegible]0.	675[illegible].	-7218.
3	[illegible]9268.	[illegible]226.	[illegible]621[illegible].	154[illegible].	-7650.
4	[illegible]58[illegible].	[illegible]212.	[illegible].	[illegible].	-723.
5	675[illegible].	1[illegible]4[illegible]2.	[illegible].	[illegible].	[illegible]435.
6	-7218.	-7650.	-723.	[illegible]435.	15123.
7	-177[illegible]8.	-[illegible]213.	-22[illegible].	-6696.	9435.
8	-22[illegible].	-[illegible].	-35482.	-22466.	-723.
9	-211[illegible].	-[illegible]41[illegible]4.	-[illegible]64.	-26213.	-7650.
10	-128[illegible].	-21124.	-[illegible]2[illegible]15.	-178[illegible]5.	-7225.

TABLE 2. DEFLECTIONS OF A TWO-HINGED ARCH.

	X1	X2	X3	X4	X5	X6
1	.[illegible]767	.[illegible]457	-.1260[illegible]	.0[illegible]5237	-.002260	.002975
2	.3114[illegible]7	.71[illegible]	.7510[illegible]	-.[illegible]	.[illegible]7263	.[illegible]7[illegible]19
3	.27[illegible]	-.1[illegible]	.[illegible]211[illegible]	.[illegible]	-.[illegible]0932	.015591
4	.17[illegible]7[illegible]	-.[illegible]156	.[illegible]71[illegible]	.[illegible]2467	.[illegible]21[illegible]	.013526
5	.[illegible]	-.[illegible]274	.[illegible]14[illegible]	.[illegible]2[illegible]1	.[illegible]7704	.024146
6	-.[illegible]	.6567	.[illegible]1[illegible]	.[illegible]146[illegible]5	.[illegible]4094	.[illegible]46549
7	-.1796[illegible]1	.11[illegible]7	-.[illegible]	.[illegible]151[illegible]	.[illegible]6594	.017741
8	-.[illegible]12[illegible]4	.1[illegible]507	-.[illegible]14[illegible]	.[illegible]14[illegible]6	.[illegible]7651	.[illegible]500[illegible]
9	-.186[illegible]	.1173[illegible]	-.[illegible]	.[illegible]	.[illegible]3646	.[illegible]8939
10	-.1[illegible]6[illegible]	.[illegible]674[illegible]	-.[illegible]7[illegible]	.[illegible]6[illegible]7	.[illegible]239[illegible]	.[illegible]2165
11	-.[illegible]2[illegible]	.[illegible]164	-.[illegible]21[illegible]	.[illegible]	-.[illegible]33	.[illegible]352

TABLE 3. [illegible] FORCES IN THE VERTICAL TRUSS MEMBERS.

	M1	M2	M3	M4	M5	M6
1	[illegible]	[illegible]	.1[illegible]	[illegible]	[illegible]	[illegible]
2	-2.29[illegible]	2.[illegible]	.55[illegible]	.44[illegible]	-.04806	-.311[illegible]
3	-3.[illegible]	1.48[illegible]	2.29[illegible]	[illegible]	.17413	-.46679
4	-2.29[illegible]	.64[illegible]	.71[illegible]	1.06[illegible]	.51918	-.26531
5	-.59129	-.10558	-.06[illegible]	.56279	1.48008	.229[illegible]
6	.08643	-.8[illegible]	-.62[illegible]	-.32176	.25719	1.1[illegible]
7	2.12314	-1.14[illegible]	[illegible]	-.47776	-.39482	.270[illegible]
8	2.59153	-1.22178	-.84[illegible]	-1.[illegible]	-.74519	-.20259
9	2.3[illegible]772	-.96461	-.71506	-.81416	-.71869	-.41528
10	1.12675	-.54967	-.40774	-.31[illegible]	-.43018	-.27650
11	.[illegible]3321	-.[illegible]127[illegible]	-.00[illegible]	[illegible]	-.[illegible]1004	-.006[illegible]

TABLE 4. MOMENT INFLUENCE LINES FOR THE GIRDER.

	M2	M3	M4	M5	M6
1	.71401	.[illegible]	.0[illegible]	-.04413	-.07221
2	15.[illegible]1524	9.[illegible]	.704[illegible]	-.26343	-3.241[illegible]
3	7.3[illegible]	21.[illegible]	1[illegible]	1.[illegible]	-4.57694
4	1.38416	9.[illegible]	[illegible].07[illegible]	7.26461	-4.[illegible]7534
5	-2.37[illegible]	.[illegible]	[illegible].41[illegible]	17.01767	4.2[illegible]166
6	-4.40365	-6.32741	[illegible]	[illegible].87617	15.68102
7	-5.25406	-9.[illegible]	-[illegible].0[illegible]	-4.56[illegible]24	4.23094
8	-4.[illegible]	-1[illegible]	-1[illegible].9707[illegible]	[illegible]	-2.43961
9	-3.7[illegible]343	-8.16[illegible]	-[illegible].477[illegible]	[illegible]	-4.42793
10	-1.92229	-4.44[illegible]	-5.31[illegible]	-4.06400	-3.27839
11	-.[illegible]437	-.10531	-.12[illegible]	-.12623	-.09227

TABLE 5. MOMENT INFLUENCE LINES FOR THE ARCH.

APPENDIX B

	δ2	δ3	δ4	δ5	δ6
2	[illegible]	21915.	26?5?.	[illegible]	24117.
3	[illegible]	44140.	[illegible]	72660.	58575.
4	[illegible]	66375.	[illegible]	118?02.	117072.
5	[illegible]	78380.	[illegible]	[illegible]	151213.
6	24117.	66375.	178273.	[illegible]	166992.
7	19193.	[illegible]	[illegible]	128427.	151213.
8	13238.	47055.	[illegible]	[illegible]	113273.
9	7325.	[illegible]	[illegible]	52646.	66277.
10	2562.	7325.	[illegible]	19193.	24117.

TABLE 1. DEFLECTIONS OF A CONTINUOUS GIRDER.

	δ2	δ3	δ4	δ5	δ6
2	2427.	10226.	10628.	6693.	-7277.
3	10226.	44151.	26117.	15325.	-7760.
4	21629.	36117.	28660.	22935.	-872.
5	6693.	15325.	22035.	22747.	9264.
6	-7277.	-7760.	-872.	9264.	14944.
7	-17854.	[illegible]	-12607.	-3852.	9264.
8	-22857.	[illegible]	-15603.	-22607.	-872.
9	-21154.	-24260.	[illegible]	-26316.	-7760.
10	-12866.	-21114.	-22857.	-17854.	-7277.

TABLE 2. DEFLECTIONS OF A TWO-HINGED ARCH.

	X2	X3	X4	X5	X6
2	.7384	.5436	-.0?482	.00770	.00271
3	-.17119	[illegible]	.00372	.012?7	.00637
4	-.1 126	.?3469	.02061	.01704	.00937
5	-. 1 1	[illegible]	.00698	.03465	.01334
6	. 74.9	-. ?128	.01414	.00951	.03667
7	.13211	[illegible]	.01634	.00715	.01330
8	.15294	-. 2333	.01963	.00543	.00932
9	.13259	-. 2229	.0151	.00341	.00632
10	. 7581	-. 1336	[illegible]	.00168	.00269

TABLE 3. REDUNDANT FORCES IN THE VERTICAL TRUSS MEMBERS.

	M1	M2	M3	M4	M5	M6
2	-3.23943	2.66[illegible]	.71[illegible]25	.41[illegible]	-.05308	-.28046
3	-3.55386	1.76237	1.94301	.96570	.10003	-.40370
4	-2.34936	.73123	.77408	1.857[illegible]	.55891	-.22840
5	-.60162	-.18433	-.06995	.57[illegible]4	1.41933	.30811
6	1.06813	-.87564	-.59680	-.35637	.30891	1.25937
7	2.26445	-1.27424	-.84949	-.[illegible]	-.40379	.30862
8	2.753[illegible]6	-1.34995	-.86486	-1.[illegible]	-.73532	-.22800
9	2.44148	-1.11043	-.6[illegible]14	-.[illegible]	-.71690	-.40342
10	1.41272	-.61505	-.3[illegible]	-.[illegible]	-.42670	-.28033

TABLE 4. MOMENT INFLUENCE LINES FOR THE GIRDER.

	M2	M3	M4	M5	M6
2	15.2[illegible]425	9.66924	3.91627	-.27926	-3.29308
3	6.86734	21.62[illegible]36	10.[illegible]	1.63495	-4.64138
4	1.19528	9.25[illegible]16	20.110[illegible]	7.17433	-2.44308
5	-2.41870	-.02610	6.34943	17.02406	4.16764
6	-4.38596	-6.25459	-3.344[illegible]	3.87988	15.55941
7	-5.[illegible]3587	-9.52676	-9.0[illegible]	-4.57948	4.16722
8	-4.64159	-1[illegible].[illegible]4143	-1[illegible].[illegible]	-8.51095	-2.44348
9	-3.46358	-8.14536	-9.3[illegible]	-6.34913	-4.64172
10	-1.79511	-4.45174	-5.2[illegible]	-4.97522	-3.29322

TABLE 5. MOMENT INFLUENCE LINES FOR THE ARCH.

APPENDIX C

	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]6
1	-.[illegible]	.[illegible]	.[illegible]	.[illegible]	.[illegible]	.[illegible]
[illegible]	-5.[illegible]	1.74[illegible]	.8[illegible]	.[illegible]	-.[illegible]81[illegible]	-.26372
[illegible]	-5.41[illegible]	.7[illegible]	2.10[illegible]	.[illegible]	.11[illegible]	-.[illegible]
[illegible]	-3.4[illegible]	.1[illegible]	.8[illegible]	1.[illegible]	.[illegible]	-.20[illegible]
[illegible]	-.77[illegible]	-.[illegible]	-.0[illegible]	.[illegible]	1.43[illegible]	.3[illegible]
6	1.76[illegible]	-.[illegible]	-.[illegible]	-.[illegible]	.[illegible]	1.29067
7	3.54[illegible]	-.57[illegible]	-.91[illegible]	-.81[illegible]	-.[illegible]	.33380
[illegible]	4.21[illegible]	-.51[illegible]	-.9[illegible]	-.[illegible]	-.71[illegible]	-.20889
[illegible]	3.65[illegible]	-.[illegible]	-.74174	-.[illegible]	-.68[illegible]	-.[illegible]236
1[illegible]	2.[illegible]	-.1[illegible]	-.[illegible]	-.44[illegible]	-.32704	-.26372
11	.[illegible]	.[illegible]	.[illegible]	.[illegible]	.[illegible]	.[illegible]

TABLE 1. MOMENT INFLUENCE LINES FOR THE GIRDER.

	[illegible]2	[illegible]	[illegible]4	[illegible]5	[illegible]6
1	.[illegible]	.[illegible]	.0[illegible]4	.0[illegible]5	.[illegible]6
[illegible]	14.[illegible]	[illegible]	[illegible]	-.42264	-3.16462
[illegible]	6.46[illegible]	2[illegible].4[illegible]7	[illegible].77[illegible]	1.[illegible]7	-4.[illegible]
[illegible]	.86[illegible]	6.41[illegible]	1[illegible].6[illegible]	6.[illegible]	-2.[illegible]666
[illegible]	-2.57[illegible]4	-.38[illegible]	6.0[illegible]	16.7[illegible]	4.00557
6	-4.[illegible]	-6.[illegible]	-[illegible].41[illegible]	3.73422	15.36[illegible]3
7	-4.[illegible]	-[illegible].[illegible]1414	-[illegible]	-4.[illegible]	4.00557
[illegible]	-4.[illegible]78	-[illegible].15[illegible]	-1[illegible]	-[illegible].[illegible]1160	-2.50666
[illegible]	-[illegible].15[illegible]	-7.2[illegible]	-[illegible].8[illegible]	-8.014[illegible]4	-4.58[illegible]
1[illegible]	-1.56[illegible]	-[illegible].[illegible]1407	-4.8[illegible]	-4.64[illegible]	-3.16462
[illegible]	.[illegible]5	.[illegible]6	.0[illegible]	.[illegible]6	.[illegible]6

TABLE 2. MOMENT INFLUENCE LINES FOR THE ARCH.

STRUCTURAL ANALYSIS OF A GIRDER-ARCH BRIDGE

by

DICK T. LEE

B.S., National Taiwan University, 1964

AN ABSTRACT OF A MASTER'S REPORT

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ABSTRACT

The consistent deformation method and the energy method are used in this report for the stress analysis of a girder-arch bridge. Using the first method, two solutions were calculated, the first of which takes into account the axial stress effects in the structure, and the second ignores these effects. A third solution was developed using the energy method.

Of the many possible methods of analysis for this type of bridge, this report presented two which may be used to analyse high-order statically indeterminate structures. A digital computer was available and therefore the calculations for both of these methods were wholly performed by the digital computer. Since the bridge analysed in this report has variable cross sections in the arch, this report shows a general procedure for a computer analysis to handle a structure with variable cross sections.

Numerical examples are given to illustrate the procedures for finding the influence lines for the structure. Since three solutions were accomplished for the same example, a comparison of the results of the various methods can be made. The comparison of the three solutions indicates that the results are reasonably close to each other.